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On magnetohydrodynamic flow in rectangular ducts: an extension of the Hunt–Stewartson approach

By T. Z. FAHIDY
Department of Chemical Engineering, University of Waterloo, Waterloo, Ontario, Canada

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The Hunt–Stewartson technique of estimating fluid velocity and magnetic flux profiles in rectangular ducts is generalized for the entire secondary boundary layer.

1. Introduction

In the recent literature on MHD flow in rectangular ducts with a uniform magnetic field imposed parallel to the electrode walls (Hunt & Stewartson 1965; Chiang & Lundgren 1967), boundary-layer theory is employed to obtain approximate analytical fluid velocity and magnetic flux profiles, in the case of a strong magnetic field. The former approach, introduced by Hunt & Stewartson, allows such estimation in the core as well as in various boundary layers associated with the cell geometry. Of these, particularly important is the boundary layer adjacent to the electrodes (secondary boundary layer), where the profiles are characterized as

$$v_s = \tfrac{1}{2}[X(\eta)+X(-\eta)], \tag{1.1}$$

$$h_s = \tfrac{1}{2}[X(\eta)-X(-\eta)]. \tag{1.2}$$

v_s and h_s are the undimensionalized velocity and induced field strength in the layer. $X(\eta)$ is expressed as

$$X = -\frac{1}{M}\operatorname{erfc}\frac{(c-\xi)M^{\frac{1}{2}}}{2(1-\eta)^{\frac{1}{2}}}+\frac{c-\xi}{2(\pi M)^{\frac{1}{2}}}\int_{\eta}^{1}\frac{\alpha(\lambda)}{(\lambda-\eta)^{\frac{3}{2}}}\exp-\left(\frac{M(c-\xi)^2}{4(\lambda-\eta)}\right)d\lambda, \tag{1.3}$$

which is their equation (2.33). Furthermore,

$$\alpha(\eta) = 1-f(\eta)\int_{1}^{\infty}\frac{s^2\,ds}{(s^4+\psi)(s^4-1)^{\frac{1}{2}}}, \tag{1.4}$$

where
$$f(\eta)\equiv\frac{8(1-\eta^2)^{\frac{1}{4}}}{\pi(1+\eta)},\quad \psi\equiv\frac{1-\eta}{1+\eta}.$$

Hunt & Stewartson suggested that (1.4) may be expressed in terms of some Gauss hypergeometric function, and confined their analysis to the limiting cases: $\eta\to 0$, $\eta\to 1$, $(c-\xi)M^{\frac{1}{2}}\gg 1$. They obtained an analytical estimate of the flux deficit pertinent to MHD pump and flow meter applications. In the following, the Hunt–Stewartson approach for the secondary boundary layer is generalized in order to allow estimation of v_s and h_s everywhere within the layer.

2. A general solution of equation (1.4)

Substitution of the variable $x \equiv s^4$ into the integral in (1.4) yields (3),

$$I = \frac{1}{4}\int_1^\infty x^{-\frac{3}{4}}(x+\psi)^{-1}(x-1)^{-\frac{1}{2}}\,dx = \tfrac{1}{4}B(\tfrac{3}{4},\tfrac{1}{2})\,{}_2F_1(1,\tfrac{3}{4};\tfrac{5}{4};-\psi), \tag{2.1}$$

except at $\psi = 0\ (\eta = 1)$. One linear transformation yields the equivalent form (4),

$${}_2F_1(1,\tfrac{3}{4};\tfrac{5}{4};-\psi) = \frac{1}{(1+\psi)^{\frac{1}{2}}}\,{}_2F_1[1,\tfrac{1}{2};\tfrac{5}{4};\tfrac{1}{2}-\tfrac{1}{2}(1+\psi)^{\frac{1}{2}}].$$

On account of the fundamental integral theorem for hypergeometric functions,

$$I = \tfrac{1}{4}B(\tfrac{3}{4},\tfrac{1}{2})\frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{1}{4})}\int_0^1 (1-t)^{-\frac{3}{4}}(1-tz)^{-\frac{1}{2}}\,dt, \tag{2.2}$$

$$z \equiv \tfrac{1}{2}-\tfrac{1}{2}(1+\psi)^{\frac{1}{2}}.$$

The integral in (2.2) can be expressed in terms of elliptic integrals (5). Upon simplification and rearrangement,

$$\alpha = 1-\frac{(2\pi)^{\frac{1}{2}}}{6}\frac{\Gamma(\frac{7}{4})}{\Gamma(\frac{5}{4})}f(\eta)\,\phi(k), \tag{2.3}$$

where $$\phi(k) \equiv \frac{K(k)-F(\varphi,k)}{[k(1-z)]^{\frac{1}{2}}},\quad k^2 \equiv -\frac{z}{1-z},\quad \varphi \equiv \cos^{-1}k^{\frac{1}{2}}.$$

Equation (2.3) is the general solution for α in the type of duct under consideration.

3. Computation of X

The integral in (1.3) now implies two separate integrations, of which the first may be performed analytically, so as to arrive at an incomplete gamma function expressible in terms of the error function. The second integral cannot be solved, analytically, but it may be reduced to a simpler form near terminal values of η. Putting $\nu^2 = (c-\xi)^2M$,

$$I(\nu,\eta) \equiv \int_\eta^1 \frac{\alpha(\lambda)\exp\left[-\dfrac{\nu^2}{4(\lambda-\eta)}\right]d\lambda}{(\lambda-\eta)^{\frac{3}{2}}} = \pi^{\frac{1}{2}}\operatorname{erfc}\left[\frac{\nu^2}{4}\frac{1}{1-\eta}\right]^{\frac{1}{2}} - 0{\cdot}418C(\nu,\eta), \tag{3.1}$$

where $$C(\nu,\eta) \equiv \int_\eta^1 \frac{f(\lambda)\,\phi(\lambda)\exp\left(-\dfrac{\nu^2}{4(\lambda-\eta)}\right)}{(\lambda-\eta)^{\frac{3}{2}}}\,d\lambda.$$

The integral $C(\nu,\eta)$ may conveniently be computed *via* usual quadrature methods for specified values of η and ν. Hence, the fields $v_s(\xi,\eta)$ and $h_s(\xi,\eta)$ can numerically be established when the Hartmann number (i.e. the strength of the external magnetic field) is known.

4. Discussion

As indicated in §2, a solution to (1.4) can be sought *via* either equivalent form of the hypergeometric function due to linear transformation. The original form ${}_2F_1(1,\frac{3}{4};\frac{5}{4};-\psi)$ violates the upper terminal condition $\alpha \to 1$, $\eta \to 1$, and obeys only the lower terminal condition $\alpha \to 0, \eta \to 0$. It is useful, however, in offering an asymptotic expansion for a small neighbourhood of η about zero. Since

$$\lim_{\eta \to 0} \alpha(\eta) = 1 - \frac{4}{3(2\pi)^{\frac{1}{2}}} \frac{\Gamma(\frac{7}{4})}{\Gamma(\frac{5}{4})} K\left(\frac{1}{(2)^{\frac{1}{2}}}\right) \lim_{\eta \to 0} (1-\eta^2)^{\frac{1}{4}},$$

$$\alpha \cong 1-(1-\eta^2)^{\frac{1}{4}}, \quad (0 \leqslant \eta < 0{\cdot}15). \tag{4.1}$$

Equation (4.1) is also deducible from the Hunt–Stewartson analysis. The linear transform equivalent ${}_2F_1[1,\frac{1}{2};\frac{5}{4};\frac{1}{2}-\frac{1}{2}(1+\psi)^{\frac{1}{2}}]$ yields negative values of α for small values of η and should not be used below $\eta < 0{\cdot}13$.

At the upper terminal value of η, as $\eta \to 1$, $\phi(k) \to 1$ and

$$\alpha \cong 1 \frac{2^{\frac{5}{4}}}{3(\pi)^{\frac{1}{2}}} \frac{\Gamma(\frac{7}{4})}{\Gamma(\frac{5}{4})} (1-\eta)^{\frac{1}{4}} = 1 - 1{\cdot}05(1-\eta)^{\frac{1}{4}}, \tag{4.2}$$

which is, again, in agreement with the Hunt–Stewartson analysis. As a point of interest, $\alpha(\eta = 0)$ may directly be obtained *via* (2.1) regardless of asymptotic considerations. Since, at $\eta = 0$, $\phi = 1$,

$$I = \tfrac{1}{4}B(\tfrac{3}{4},\tfrac{1}{2})\,{}_2F_1(1,\tfrac{3}{4};\tfrac{5}{4};-1) = \tfrac{1}{4}B(\tfrac{3}{4},\tfrac{1}{2})\frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})} = \frac{\pi}{8}.$$

Hence,
$$\lim_{\eta \to 0} \alpha(\eta) = 1 - \lim \frac{8}{\pi}\frac{(1-\eta^2)^{\frac{1}{4}}}{1+\eta}\frac{\pi}{8} = 0.$$

In figure 1 the above discussion is summarized. Consequently, when $\eta \to 1$,

$$I(\nu,\eta) \cong \frac{\exp\left[-\dfrac{\nu^2}{4}\dfrac{1}{1-\eta}\right]}{\left[\dfrac{\nu^2}{4}\dfrac{1}{1-\eta}\right]^{\frac{1}{2}}} - 0{\cdot}418\, C_1(\nu,\eta), \tag{4.3}$$

where
$$C_1(\nu,\eta) \equiv \frac{f\left(\dfrac{1+\eta}{2}\right)\phi\left(\dfrac{1+\eta}{2}\right)\exp\left(-\dfrac{\nu^2}{4\{\frac{1}{2}(1-\eta)\}}\right)}{[\frac{1}{2}(1-\eta)]^{\frac{3}{2}}}(1-\eta),$$

and, when $\eta \to 0$,

$$I(\nu,\eta) \simeq \pi^{\frac{1}{2}}\,\mathrm{erfc}\left(\frac{\nu^2}{4}\right)^{\frac{1}{2}} - \int_0^1 \frac{(1-\lambda)^{\frac{1}{4}}\exp(-\nu^2/4\lambda)\,d\lambda}{\lambda^{\frac{3}{2}}}. \tag{4.4}$$

Equations (4.3) and (4.4) may further be simplified for large values of ν (large magnetic field strength), but the asymptotic formula of Hunt & Stewartson (2.39) is more compact for numerical calculations.

The Hunt–Stewartson approach and its present extension do not adequately describe the $v(\xi,\eta)$ fields at Hartmann numbers below $M = 15$ (Hunt, private communication). The only known analytical solution of the problem for any arbitrary Hartmann number is that of Grinberg (1961), in terms of generalized Green functions. In weak magnetic fields, the Grinberg approach is still burdened by a large number of numerical integrations and matrix inversions, in spite of substantial simplifications of the general solution. In medium and

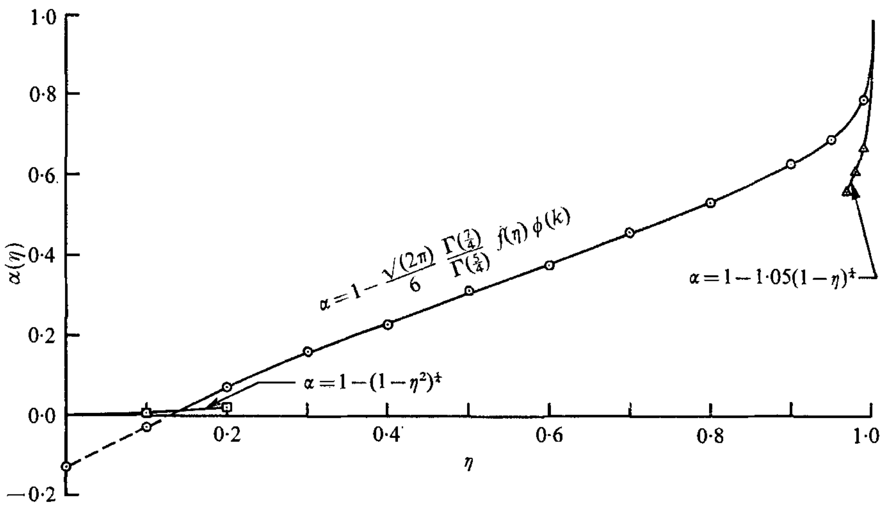

FIGURE 1. The function $\alpha(\eta)$ *versus* η in the domain $0 \leqslant \eta \leqslant 1$.

strong magnetic fields the hereby extended approach permits computation of the velocity/magnetic flux fields in a relatively simple numerical fashion aided by recent extensive tables on complete and incomplete elliptic integrals (Belyakov *et al.* 1965).

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